

EXAM SETS & NUMBERS (PART 1: SETS),  
February 3rd, 2023, 3:00pm–5:00pm,  
Aletta Jacobshal 01 H12-J3.

*Write your name on every sheet of paper that you intend to hand in.*

*Please provide **complete** arguments for each of your answers. This part of the exam consists of 2 questions. You can score up to 9 points for each question, and you obtain 2 points for free. In this way you will score in total between 2 and 20 points.*

- (1)
  - (a) [2 points] Write down all elements of the set  $(2^{\{0,1\}} \times \emptyset) \setminus \{0, 1\}$ .
  - (b) [2 points] Explicitly write down three subsets of  $\mathbb{R}$  called  $A, B, C$  such that  $A \cap B$  and  $B \cap C$  and  $C \cap A$  are non-empty but  $A \cap B \cap C = \emptyset$ .
  - (c) [2 points] Define a function  $f : \{x, y, z\} \rightarrow \{0, 1, 2\}$  by  $f(x) = 0$ ,  $f(y) = 2$  and  $f(z) = 0$ . Write down the domain, co-domain and range for  $f$  and compute  $f^{-1}(\{0, 2\})$ .
  - (d) [1+2 points] If the function  $f$  from the previous part injective? Does there exist a function  $g : \{0, 1, 2\} \rightarrow \{x, y, z\}$  such that  $g \circ f$  is a bijection? Give an example of such a function  $g$  or prove that it cannot exist.
  
- (2) Define a set  $M = \{a, b, r, c, d\}$  and a function  $F : M \rightarrow M$  by  $F(a) = b$ ,  $F(b) = r$ ,  $F(r) = a$  and  $F(c) = c$  and  $F(d) = d$ .
  - (a) [1 point] Find  $F^{(3)}(a)$  and  $F(\{a, b, r, a, c, a, d, a, b, r, a\})$ .
  - (b) [2 points] Prove by induction that for all  $n \in \mathbb{N}$  we have  $F^{(3n)} = \text{id}_M$ .
  - (c) [2 points] Does the following define an equivalence relation  $\sim$  on  $M^2$ ? Define  $(x, y) \sim (z, w)$  iff  $F(\{x, y\}) = F^{-1}(\{z, w\})$ .
  - (d) [2 points] Prove that there exist infinitely many functions with domain  $\mathbb{N}$  and co-domain  $M$ .
  - (e) [2 points] Suppose  $R$  is an equivalence relation on  $M$ . Prove that if  $(x, y) \in R$  and  $(x, z) \in R$  and  $(z, w) \in R$  then also  $(w, y) \in R$ .

**If you are only retaking the sets part, this is the side you need to complete. Otherwise, please turn over for part 2 on numbers and do that part on a DIFFERENT piece of paper.**

EXAM SETS & NUMBERS (PART 2: INTEGERS AND MODULAR ARITHMETIC),  
February 3rd, 2023, 3:00pm–5:00pm,  
Aletta Jacobshal 01 H12-J3.

*Write your name on every sheet of paper that you intend to hand in.*

*Please provide **complete** arguments for each of your answers. This part of the exam consists of 2 questions. You can score up to 9 points for each question, and you obtain 2 points for free. In this way you will score in total between 2 and 20 points.*

- (1) A question regarding divisors and multiples: given  $a \in \mathbb{Z}$ , the multiples of  $a$  are the integers contained in  $a \cdot \mathbb{Z} = \{m \in \mathbb{Z} : a|m\}$ . For any triple  $a, b, c \in \mathbb{Z}$  we denote

$$a \cdot \mathbb{Z} + b \cdot \mathbb{Z} + c \cdot \mathbb{Z} = \{k + \ell + m : k \in a \cdot \mathbb{Z}, \ell \in b \cdot \mathbb{Z}, m \in c \cdot \mathbb{Z}\}.$$

- (a) [2 points] Show that  $42 \notin 294 \cdot \mathbb{Z} + 42042 \cdot \mathbb{Z} + 42^3 \cdot \mathbb{Z}$ .  
(b) [2 points] What is the least positive element of the intersection

$$84 \cdot \mathbb{Z} \cap 126 \cdot \mathbb{Z} \cap 210 \cdot \mathbb{Z}?$$

(no credit is given for an answer without a solid explanation).

- (c) [2 points] Show that if all of  $a, b, c \in \mathbb{Z}$  are nonzero, then  $\gcd(a, b)$  and  $\gcd(a, c)$  and  $\gcd(b, c)$  are elements of  $a \cdot \mathbb{Z} + b \cdot \mathbb{Z} + c \cdot \mathbb{Z}$ .  
(d) [1+2 points] Prove that

$$|a| = |b| = |c| \iff a \cdot \mathbb{Z} + b \cdot \mathbb{Z} + c \cdot \mathbb{Z} = a \cdot \mathbb{Z} \cap b \cdot \mathbb{Z} \cap c \cdot \mathbb{Z}.$$

- (2) This problem considers, for integers  $k \in \mathbb{Z}_{\geq 0}$ , the numbers

$$a_k = 2 \cdot 3^k + 4 \cdot 3^{2k}$$

(and some modular arithmetic).

- (a) [1 point] Explain why  $a_{k+2} = 12a_{k+1} - 27a_k$  for every  $k \geq 0$ .  
(b) [2 points] Prove that  $3^{4\ell} \equiv (1 + 80\ell) \pmod{100}$  whenever  $\ell \geq 0$  and that  $a_k \equiv 6 \pmod{100}$  for every  $k \equiv 0 \pmod{4}$ .  
(c) [2 points] Explain why the sequence  $(a_k \pmod{100})_{k \geq 0}$  of elements in  $\mathbb{Z}/100\mathbb{Z}$  is periodic.  
(d) [2 points] Determine  $b$  with  $0 \leq b \leq 99$  such that  $a_{3022023} \equiv b \pmod{100}$ .  
(e) [2 points] The next ‘prime’ date is 11 – 02 – 2023. In other words  $p = 11022023$  is a prime number. Given this  $p$ , does  $m \in \mathbb{Z}$  with  $0 \leq m \leq 10$  exist such that  $a_{p-1} \equiv m \pmod{p}$ ?

**If you are only retaking the numbers part this side is all you need to complete, otherwise please turn over for part 1 on sets and do that part on a DIFFERENT piece of paper.**